Structure Theorems for Signed Graphs

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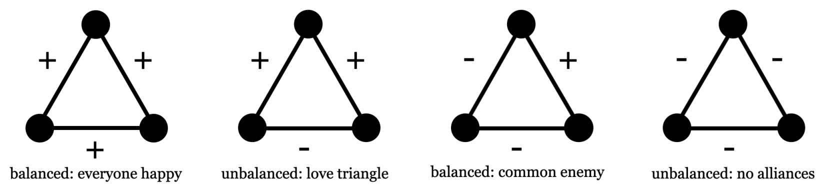
June 2025

1 Introduction

The *signed graph* is an object that combines graph theory and social science to arrive at interesting conclusions. We assign each of a simple unweighted undirected graph with either a + or − sign, and we are able to derive structural theorems based on these signs. From [CH56] we find that this simple addition can model social situations of friendship and enmity. Imagine each vertex as a person, a + edge as a positive relationship and a − edge as a negative relationship. A signed graph is *balanced* if the product of edge signs along every cycle is positive, a notion that extends to balance in social situations.



In the following theorems, we observe the behavior of a signed graph as a complete social system.

2 Harary's Theorems

A signed graph (G, σ) is a graph $G = (V, E)$ with an assignment of signs $\sigma : E \to \{-1, +1\}$. We call an edge *positive* or *negative* if its sign is $+1$ or -1 respectively. The sign of a path is the product of the path's edges. By Harary [Har53] we get the following theorems.

Definition. A signed graph is *balanced* if every cycle in the graph has a positive sign.

Theorem 1. *For every complete signed graph $G = (K_n, \sigma)$, G is balanced if and only if there is a cut S, T such that all internal edges $S \leftrightarrow S$, $T \leftrightarrow T$ are positive and all cross edges $S \leftrightarrow T$ edges are negative.*

A social interpretation of this theorem is that a balanced social system consists of two tightly knit cliques who oppose each other.

Proof.

$\rightarrow$ Direction: Suppose G is balanced. Let v be an arbitrary vertex in G. Let $X \subseteq V$ be the set of vertices connected to v by a positive edge and v itself, and let Y be the set of all other vertices (i.e., vertices connected to v by a negative edge since G is the complete graph.) It is clear that X, Y are disjoint and $X \cup Y = V$.

Any pair of vertices $x_1, x_2 \subseteq X$ must have a positive edge between them. If one of them is v, then this is true by construction. Otherwise, there is a cycle $v \to x_1 \to x_2$ that must be positive since G is balanced. Since (v, x_1) and (v, x_2) are positive by definition, it follows that (x_1, x_2) must be positive as well.

Any pair of vertices $y_1, y_2 \subseteq Y$ must have a positive edge between them. Otherwise, there is a cycle $v \to y_1 \to y_2$ that must be positive since G is balanced. Since (v, y_1) and (v, y_2) are negative by definition, it follows that (y_1, y_2) must be positive. This satisfies the theorem.

$\leftarrow$ Direction: Suppose there is a cut S, T that satisfies the theorem. Since any cycle must have an even number of edges crossing the cut, the cycle will be positive. $\square$

Lemma 2. *Every subgraph of a balanced signed graph is also balanced.*

Proof. Every cycle in the subgraph corresponds to a cycle in the original graph, so it must be positive. $\square$

Theorem 3. *A signed graph is balanced if and only if for all pairs of distinct vertices u, v all paths between u and v have the same sign.*

Proof.
$\rightarrow$ Direction: Suppose G is balanced. Consider any two paths α_1, α_2 connecting u and v. Deleting the common edges from α_1 and α_2 (if any) yields a collection of edge-disjoint cycles. Let z be an arbitrary cycle from this collection. z is comprised of a subset of edges from α_1 and a subset from α_2. Since G is balanced, z is a positive cycle, so the subsets α_1 and α_2 must have the same sign. Since each subset shares the same sign, and the common edges are common to both paths, it follows that α_1 and α_2 must have the same sign.

$\leftarrow$ Direction: Suppose all paths between any pair of vertices u, v have the same sign. Then any cycle containing u and v must be positive. Since u and v are arbitrary, all cycles must be positive. $\square$

We are now ready to extend Theorem 1 to general signed graphs.

Theorem 4. *A signed graph is balanced if and only if there is a cut S, T such that all internal edges $S \leftrightarrow S$, $T \leftrightarrow T$ are positive and all cross edges $S \leftrightarrow T$ edges are negative.*

Proof.
$\rightarrow$ Direction: Suppose G is balanced, and without loss of generality that G is connected. Let u, v be a pair of vertices that are not connected by an edge. By Theorem 3, all paths between u and v must have the same sign. If we add an edge (u, v) with this sign, all new cycles introduced are positive, and G remains balanced. If we continue this process until G is the complete graph, the theorem follows from Theorem 1.

$\leftarrow$ Direction: Suppose a cut S, T exists. For each pair of vertices u, v that are not connected by an edge, add a positive edge between them if they are both in S or both in T. Otherwise, add a negative edge. Once G is the complete graph, the theorem follows from Theorem 1. $\square$

From Harary in [HK80] we also get a simple algorithm for testing balance.

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Let G⁺ = (V, E⁺) be the subgraph containing only +1 edges
Find the connected components C_1, C_2, ..., C_k of G⁺
foreach component C_i do
    if there exists a -1 edge between two nodes in C_i then
        Output Unbalanced; return
    end
end
Construct graph H by collapsing each C_i into a single node and adding -1 edges between
  components
Replace any multi-edges in H with a single edge
if H is bipartite then
    Output Balanced
end
else
    Output Unbalanced
end
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3 Random Signed Graphs

Now we explore a model of *random* signed graphs $G_{n,p,q}$ that closely follows the Erdős–Rényi model. Let p, q be fixed with $0 < p+q < 1$. Given a set of n vertices, between each pair of distinct vertices x and y there is either a positive edge with probability p or a negative edge with probability q, or else there is no edge at all with probability $1-(p+q)$. The edges between different pairs of vertices are chosen independently.

We present an alternate way to view $G_{n,p,q}$. Let $\tilde{G}_{n,p,q}$ be a random *unsigned* graph which has the same probability distribution as the standard random graph $G_{n,p+q}$ with edge probability $p+q$. We denote $E(\tilde{G}_{n,p,q})$ as this graph's edge set. Then, for any fixed pair of vertices x, y assign:

$$\mathbb{P}\left(\{x,y\} \text{ is positive in } G_{n,p,q} | \{x,y\} \in E(\tilde{G}_{n,p,q})\right) = \frac{p}{p+q}$$

and

$$\mathbb{P}\left(\{x,y\} \text{ is negative in } G_{n,p,q} | \{x,y\} \in E(\tilde{G}_{n,p,q})\right) = \frac{q}{p+q}$$

In other words, $G_{n,p,q}$ can be considered as the random variable on the set of the signed graphs on n vertices whose probability distribution is given by

$$\mathbb{P}(G_{n,p,q} = G) = p^m q^k (1-p-q)^{\binom{n}{2}-m-k}$$

where G is a signed graph with m positive edges and k negative edges.
From [MMM12] we get the following theorem.

Theorem 5. *Let p, q be fixed with $0 < p+q < 1$. Then $G_{n,p,q}$ is unbalanced with high probability.*

First, we will prove the following useful lemma.

Lemma 6. *Let H be a fixed set of h distinct pairs of vertices of $G_{n,p,q}$. Then*

$$\mathbb{P}\left(H \textit{ is positive in } G_{n,p,q} \mid H \subseteq E(\tilde{G}_{n,p,q})\right) = \frac{1}{2}\left[1 + \left(\frac{p-q}{p+q}\right)^h\right]$$

and

$$\mathbb{P}\left(H \textit{ is negative in } G_{n,p,q} \mid H \subseteq E(\tilde{G}_{n,p,q})\right) = \frac{1}{2}\left[1 - \left(\frac{p-q}{p+q}\right)^h\right]$$

Proof. Let

$$\begin{aligned} p_1 &= \mathbb{P}(H \text{ is positive in } G_{n,p,q} | H \subseteq E(\tilde{G}_{n,p,q})) \\ &= \sum_{i \text{ even}} \mathbb{P}(|H^-| = i) \end{aligned}$$

where $|H^-|$ is the number of negative edges in H. Then,

$$p_1 = \frac{1}{(p+q)^h} \sum_{i \text{ even}} \binom{h}{i} q^i p^{h-i}$$

Similarly, let

$$p_2 = \mathbb{P}(H \text{ is negative in } G_{n,p,q} | H \subseteq E(\tilde{G}_{n,p,q})) = \frac{1}{(p+q)^h} \sum_{i \text{ odd}} \binom{h}{i} q^i p^{h-i}$$

Then we get the following system of equations:

$$\begin{cases} p_1 + p_2 &= 1 \\ p_1 - p_2 &= \left[\frac{p-q}{p+q}\right]^h \end{cases}$$

Solving the system completes the proof of the theorem. □

Now we are ready to prove Theorem 5. Let $\mathcal{T}$ denote a maximum set of edge-disjoint triangles in the complete graph K_n. To prove the theorem, we will show that $G_{n,p,q}$ contains a negative triangle from $\mathcal{T}$ with high probability.

It is clear that $|\mathcal{T}| \geq \left\lfloor \frac{n}{3} \right\rfloor$. Let T be a fixed element of $\mathcal{T}$. We have

$$\mathbb{P}\left(T \subseteq \tilde{G}_{n,p,q} \text{ and } T \text{ is negative}\right) = \mathbb{P}\left(T \text{ is negative} \mid T \subseteq \tilde{G}_{n,p,q}\right) \times \mathbb{P}\left(T \subseteq \tilde{G}_{n,p,q}\right)$$

Using Theorem 6, we get

$$\mathbb{P}\left(T \subseteq E(\tilde{G}_{n,p,q}) \text{ and } T \text{ is negative}\right) = \frac{1}{2}\left[1 - \left(\frac{p-q}{p+q}\right)^3\right](p+q)^3$$

$$= \frac{1}{2}\left[(p+q)^3 - (p-q)^3\right]$$

Thus, the probability that $G_{n,p,q}$ contains a negative triangle from $\mathcal{T}$ is at least

$$1 - \left(1 - \frac{1}{2}\left[(p+q)^3 - (p-q)^3\right]\right)^{\left\lfloor \frac{n}{3} \right\rfloor}$$

Since p and q are fixed, this expression tends to 1 as $n \to \infty$. □

We can interpret this theorem to say that balance in social systems is very rare as the number of people grows without any predefined social structure.

References

[CH56] Dorwin Cartwright and Frank Harary. "Structural balance: a generalization of Heider's theory". In: *Psychological Review* 63.5 (1956), pp. 277–293. DOI: 10.1037/h0046049.

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[HK80] Frank Harary and Jerald A. Kabell. "A simple algorithm to detect balance in signed graphs". In: *Mathematical Social Sciences* 1.1 (1980), pp. 131–136.

[MMM12] Abdelhakim El Maftouhi, Yannis Manoussakis, and Olga Megalakaki. "Balance in Random Signed Graphs". In: *Internet Mathematics* 8.4 (2012), pp. 364–380. DOI: 10.1080/15427951.2012.719877.